Journal of Global Optimization **21:** 1–14, 2001.

Piecewise-Convex Maximization Problems ⋆

Global Optimality Conditions

IDER TSEVENDORJ
INRIA, Domaine de Voluceau, Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France (e-mail: ider.tsevendorj@inria.fr

(Received 26 September 2000; accepted in revised form 19 April 2001)

Abstract. A function $F : R^n \to R$ is called a piecewise convex function if it can be decomposed into $F(x) = \min\{f_j(x) \mid j \in M\}$, where $f_j : R^n \to R$ is convex for all $j \in M = \{1, 2..., m\}$. We consider $\max F(x)$ subject to $x \in D$. It generalizes the well-known convex maximization problem. We briefly review global optimality conditions for convex maximization problems and carry one of them to the piecewise-convex case. Our conditions are all written in primal space so that we are able to proposea preliminary algorithm to check them.

AMS subject classifications: 90C46, 90C47

Key words: Global optimization, Sufficient and necessary optimality conditions, Nonconvex and nonsmooth problem, Piecewise convex function

Abbreviations: (CMP) – Convex Maximization Problems; (PCMP) – Piecewise-Convex Maximization Problems

1. Introduction

Let D be a nonempty, compact and convex subset of $\mathbb{R}^n$ and let M be a finite index set. We begin by introducing some definitions.

DEFINITION 1. A function $F : R^n \to R$ is called a piecewise convex function if it can be decomposed into:

$$F(x) = \min\{f_j(x) \mid j \in M\}, \tag{1}$$

where $f_j : R^n \to R$ is convex for all $j \in M = \{1, 2..., m\}$.

DEFINITION 2. A problem

$$\begin{cases} \text{maximize} \quad F(x), \\ \text{subject to } x \in D \end{cases} \tag{PCMP}$$

is called a piecewise convex maximization problem, if $F(\cdot)$ is a piecewise convex function and D is a convex set.

⋆ This research was supported by Direction des transports terrestres, Ministère de l'Equipement des Transports et du Logement (projet N 226 75 01 63-41 50).

The purpose of this paper is to establish necessary and sufficient optimality conditions for the piecewise convex maximization problems (PCMP).

Such problems have many practical and theoretical applications [9], but a solution for them does not seem to have been extensively studied.

If in (1) all functions $f_j(\cdot)$ are affine and 'min' is replaced by 'max' then the problem (PCMP) turns out to be a convex maximization problem (CMP) with piecewise-linear convex function. Of course, it also simplifies to (CMP) whenever $m = 1$.

An important property of the convex maximization problem is that every local (and in particular global) solution is achieved at an extreme point of the feasible domain. In general, this property does not hold for (PCMP) as a large number of local optima could lie anywhere in D.

The present paper is organized as follows. First, we recall some local and global necessary and sufficient conditions for the convex maximization in Section 2. As a result, we derive optimality conditions for a nonsmooth convex maximization problems; in Section 3, we extend the previous section's result to (PCMP) and in Section 4, we propose a preliminary algorithm for (PCMP) based on our global optimality conditions.

2. Convex Maximization Problem

Let $D \subset \mathbb{R}^n$ be a convex and compact set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function; in this section, we consider the global optimization (*convex maximization*) problem:

$$\begin{cases} \text{maximize} \quad f(x), \\ \text{subject to } x \in D \end{cases} \tag{CMP}$$

The state-of-the-art in convex maximization including many algorithms and abundant applications, is extensively described in text books [5, 6], in papers [2, 8, 13, 14] and surveys [1].

In recent years, several interesting necessary and sufficient optimality conditions characterizing a global maximum ($z \in D$) of (CMP) have been proposed:

Strekalovsky's necessary and sufficient condition ([12])

$$\partial f(y) \subset N(D, y) \quad \text{for all } y \;\; : \;\; f(y) = f(z), \tag{SgNS}$$

Hiriart-Urruty's necessary and sufficient condition ([7])

$$\partial_\varepsilon f(z) \subset N_\varepsilon(D, z) \quad \text{for all } \varepsilon \geqslant 0, \tag{HUgNS}$$

and Flores-Bazan's necessary and sufficient condition ([3])

$$\partial_\gamma f(z) \subset \partial_\gamma \delta(\cdot|D)(z), \tag{FBgNS}$$

where $\partial f(y)$ and $N(D, y)$ are the subdifferential of a function $f(\cdot)$ and normal cone to a set D at point y, respectively

$$\partial f(y) = \{y^* \in \mathbb{R}^n \mid \quad f(x) - f(y) \geqslant \langle y^*, x - y\rangle \text{ for all } x \in \mathbb{R}^n\},$$
$$N(D, y) = \{y^* \in \mathbb{R}^n \mid \quad \langle y^*, x - y\rangle \leqslant 0 \text{ for all } x \in D\},$$

$\partial_\varepsilon f(z)$ and $N_\varepsilon(D, z)$ are the ε-subdifferential of a function $f(\cdot)$ and the set of ε-normal directions to a set D at a point z, respectively

$$\partial_\varepsilon f(z) = \{z^* \in \mathbb{R}^n \mid \quad f(x) - f(z) \geqslant \langle z^*, x - z\rangle - \varepsilon \quad \text{for all} \quad x \in \mathbb{R}^n\},$$
$$N_\varepsilon(D, z) = \{z^* \in \mathbb{R}^n \mid \quad \langle z^*, x - z\rangle \leqslant \varepsilon \text{ for all } x \in D\},$$

$\delta(\cdot|D)$ is the indicator function of D and $\partial_\gamma f(z)$ is the γ-subdifferential of a function $f(\cdot)$ at point z:

$$\partial_\gamma f(z) = \{\phi(x) \in C \mid f(x) - f(z) \geqslant \phi(x) - \phi(z) \text{ for all } x \in \mathbb{R}^n\}.$$

It is worthwhile noticing that the above the conditions generalize Rockafellar's local necessary optimality condition

$$\partial f(z) \subset N(D, z). \tag{RlN}$$

It is not difficult to see that (SgNS) with $(y = z)$, (HUgNS) with $(\varepsilon = 0)$ and (FBgNS) with (linear ϕ) all imply (RlN).

The aim of this section is to improve the (*classical*) local necessary optimality condition

$$\partial f(z) \bigcap N(D, z) \neq \emptyset \tag{ClN}$$

in order to fully describe a global maximum; the classical condition (ClN) is not sufficient even for a local maximum.

THEOREM 1 [16]. *Let $z \in D$ and assume there exists a $v \in \mathbb{R}^n$ s.t. $f(v) < f(z)$. Then a necessary and a sufficient condition for $z \in D$ to be a global maximum for (CMP) is:*

$$\partial f(y) \bigcap N(D, y) \neq \emptyset \textit{ for all } y \textit{ s.t. } f(y) = f(z) \tag{gNS}$$

Proof.

$(\Rightarrow)$ Let z solve (CMP), in other words $f(z) \geqslant f(x)$ for all $x \in D$. Then, due to convexity of the function $f(\cdot)$ and the definition of subdifferential, for all y such that $f(y) = f(z)$

$$0 \geqslant f(x) - f(z) = f(x) - f(y) \geqslant \langle y^*, x - y\rangle,$$

holds for all $y^* \in \partial f(y)$ and $x \in D$ so that $y^* \in \partial f(y) \bigcap N(D, y)$.

($\Leftarrow$) By contradiction, let z not a global maximum of (CMP). Thus, there is a

$$u \in D \text{ such that } f(u) > f(z).$$

Then, let us consider a convex combination of u and a point v such that $f(v) < f(z)$

$$y(\alpha) = \alpha v + (1-\alpha)u.$$

There is a number $\alpha_0 \in]0, 1[$ such that $f(y(\alpha_0)) = f(z)$ since $f(\cdot)$ is continuous and $f(v) < f(z) < f(u)$. Now, one shows $\partial f(y(\alpha_0)) \not\subset N(D, y(\alpha_0))$.

For all subgradients y_0^* of $f(\cdot)$ at $y(\alpha_0)$ satisfying $f(y(\alpha_0)) = f(z)$ and for $u \in D$, the following holds

$$\begin{aligned} \langle y_0^*, u - y(\alpha_0)\rangle &= \langle y_0^*, \frac{y(\alpha_0) - \alpha_0 v}{1-\alpha_0} - y(\alpha_0)\rangle \\ &\geqslant \frac{\alpha_0}{\alpha_0 - 1}(f(v) - f(y(\alpha_0))) > 0, \end{aligned}$$

proving $y_0^* \notin N(D, y(\alpha_0))$ for all $y_0^* \in \partial f(y(\alpha_0))$.

REMARK 1. The classical local necessary condition (ClN), compared to the necessary part of the global optimality condition (gNS), only considers z instead of all points on the level set $\{y \mid f(y) = f(z)\}$.

REMARK 2. When $f(\cdot)$ is a differentiable function, we retrieve Strekalovsky's condition $(SgNS)$ since $\partial f(y)$ consists of a single element $\nabla f(y)$.

REMARK 3. In the case of a nondifferentiable function $f(\cdot)$ one can see the difference between (gNS) and $(SgNS)$ since the latter is in general intractable to check (see example).

EXAMPLE 1. Consider the problem in $\mathbb{R}^2$ to maximize the piecewise linear convex function (polyhedral) defined by (see Figure 1):

$$f(x_1, x_2) = \max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

subject to

$$D = \{x \in \mathbb{R}^2 \;/\; -3 \leqslant x_i \leqslant 3,\; i = 1, 2\}.$$

- At point $z = (3, -3)^\top$, **the classical optimality condition** (ClN) is satisfied: for $z^* = (3, -1)^\top \in \partial f(z)$, $\langle z^*, x - z\rangle \leqslant 0$ holds for all $x \in D$; however, z is not local maximum since **Rockafellar's condition** (RlN) is violated (for instance, the subgradient $\overline{z}^* = (-1, -3)^\top \in \partial f(z)$, but $\overline{z}^* \notin N(D, z)$).
- The point $z' = (3, 3)^\top$ is a local maximum. But local optimality information does not allow us to decide whether it is a global maximum.

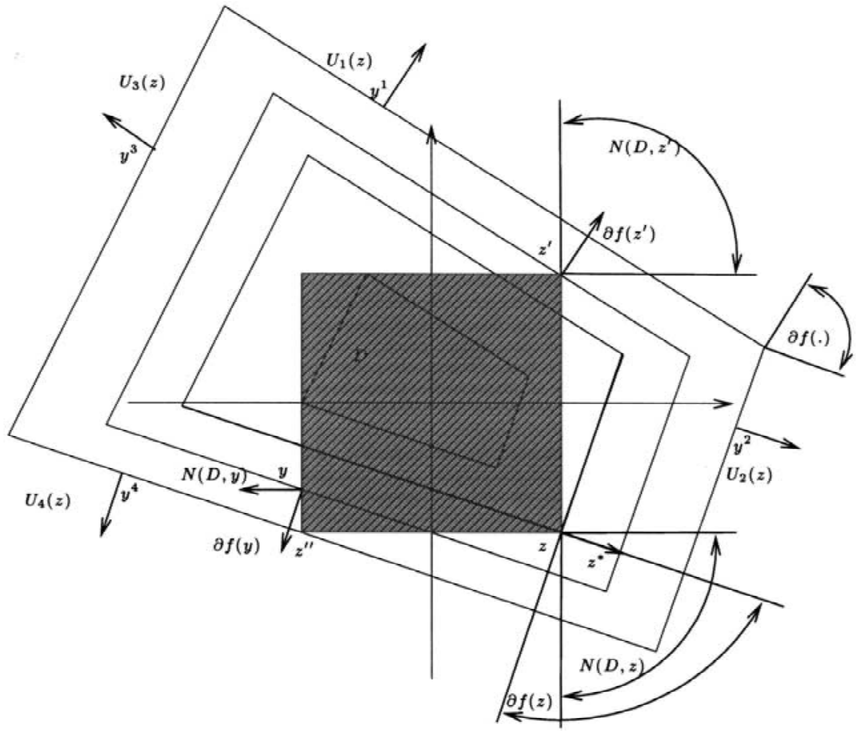

Figure 1. Example 1

Let us denote $U(z) = \{y \in \mathbb{R}^2 \ / \ f(y) = f(z)\}$ and $U(z) = U_1 \cup U_2 \cup U_3 \cup U_4$, where

$$U_1(z) = \{y \ / \ 2y_1 + 3y_2 = f(z)\},$$
$$U_2(z) = \{y \ / \ 3y_1 - y_2 = f(z)\},$$
$$U_3(z) = \{y \ / \ -2y_1 + y_2 = f(z)\},$$
$$U_4(z) = \{y \ / \ -2y_1 - 6y_2 = f(z)\}.$$

Using Theorem 1, it is easy to see that the necessary condition is violated at point $z' = (3,3)^\top$ since for all $y \in U_4(z')$ $y^* = (-1,-3)^\top \in \partial f(y)$ but $y^* \notin N(D,y)$; therefore, z' is not a global maximum.

– Now, let consider the point $z'' = (-3,-3)^\top$; in order to conclude for a global maximum, one has to check the sufficient part of the global optimality condition (gNS). According to Theorem 1, it amounts to checking whether y_i^* belongs to $N(D, y^i)$, for all $y^i \in U_i$, $i = 1,2,3,4$.
  Notice that using **Strekalovsky's condition** $(SgNS)$ instead, requires to checking in addition $y^* \in \partial f(y)$, for all $y \in U_k(z) \cap U_l(z)$ and $(k,l) \in \{(1,2),(2,3),(3,4),(4,1)\}$, which is an intractable problem in general.

3. Piecewise Convex Maximization Problem

In this section, we consider the nonconvex and nonsmooth *piecewise convex maximization problem* (also known as the discrete maxmin problem)

$$\begin{cases} \text{maximize} \quad F(x), \\ \text{subject to } x \in D \end{cases} \tag{PCMP}$$

where $D \in \mathbb{R}^n$ is a convex and compact set and $F(\cdot)$ is piecewise convex.

We will use further notations, $clco(D)$ as the closure of the convex hull of set D and :

$$I(z) = \{i \in M \ / \ f_i(z) = F(z)\},$$
$$D_k(z) = D \cap \{x \mid f_j(x) > F(z) \text{ for all } j \in M \setminus \{k\}\}$$

for respectively, a set of active functions at z, and a special subdomain.

LEMMA 1. *If for a point $z \in D$, both $F(z) \geqslant F(x)$ for all $x \in D$ and $f_k(z) = F(z)$ for some $k \in M$ hold, then $f_k(z) \geqslant f_k(x)$ for all $x \in D_k(z)$.*

Proof. Let us assume that there exists some $u \in D_k(z)$ such that $f_k(u) > f_k(z)$.

Then, from $u \in D_k(z)$ we get $f_j(u) > F(z)$ for all $j \in M \setminus \{k\}$ so that $F(u) = \min\{f_j(u) \mid j \in M\} > F(z)$, a contradiction to $F(z) \geqslant F(x)$ for all $x \in D$.

Lemma 1 together with the necessary part of (gNS) provides a necessary condition for a global solution to (PCMP).

PROPOSITION 1. *If $z \in D$ is a global maximum of (PCMP) then for all $k \in I(z)$*

$$\partial f_k(y) \bigcap N(D_k(z), y) \neq \emptyset \textit{ for all } y \textit{ s.t. } f_k(y) = F(z). \tag{gN}$$

Proof. By definition of the set $I(z)$ we have $f_k(z) = F(z)$ for all $k \in I(z) \subset M$. Then by Lemma 1, if z solves (PCMP) then z is maximum for $f_k(\cdot)$ over $D_k(z)$ for all $k \in I(z)$. Using the necessary part of theorem 1 and the definition of $I(z)$ leads to

$$\partial f_k(y) \bigcap N(D_k(z), y) \neq \emptyset \quad \text{for all} \quad y \text{ s.t. } f_k(y) = f_k(z) = F(z).$$

In order to strengthen this necessary condition and achieve a sufficient condition for (PCMP), we first prove the following two lemmas.

LEMMA 2. *Given vectors $c \in \mathbb{R}^n$ and $u \in clco(D_k(z))$, then there exists a $w \in D_k(z)$ such that $\langle c, u\rangle \leqslant \langle c, w\rangle$.*

Proof. Let us assume that there exists some $c \in \mathbb{R}^n$ and $u \in clco(D_k(z))$ such that $\langle c, u\rangle > \langle c, w\rangle$ for all $w \in D_k(z)$.

Then, by Caratheodory's theorem [10] (p. 155) $u \in clco(D_k(z))$ implies that there are $x^1, x^2, ..., x^{n+1} \in D_k(z)$ and nonnegative $\alpha_i \in \mathbb{R}$ such that $\sum_{i=1}^{n+1} \alpha_i =$

1 and $u = \sum_{i=1}^{n+1} \alpha_i x^i$. From assumption, we have $\langle c, u\rangle > \langle c, x^i\rangle$ for all $i = 1, 2, ..., n+1$. Now, multiplying the previous inequalities by corresponding α_i and summing yields

$$\sum_{i=1}^{n+1} \alpha_i \langle c, u\rangle > \langle c, \sum_{i=1}^{n+1} \alpha_i x^i\rangle = \langle c, u\rangle,$$

a contradiction.

LEMMA 3. *Given continuous functions $g(\cdot), h(\cdot)$, let $\varphi(\cdot) = \min\{g(\cdot), h(\cdot)\}$. If for all $x \in D$, $h(x) > \varphi(z) \geqslant g(x)$ for some z then $\varphi(z) \geqslant \varphi(x)$ for all $x \in D$.*

Proof. We will use the simple observation in $\mathbb{R}$ that for any a, b, c, if $a \leqslant b$ then $\min\{a, c\} \leqslant \min\{b, c\}$. Decompose D as disjoint union $D = D^+ \cup D^-$, where $D^+ = D \cap \{x \mid h(x) > \varphi(z)\}$ and $D^- = D \cap \{x \mid h(x) \leqslant \varphi(z)\}$.

For $x \in D^+$, from $g(x) \leqslant \varphi(z)$ one has $\min\{g(x), h(x)\} \leqslant \min\{\varphi(z), h(x)\}$ and hence $\varphi(z) \geqslant \varphi(x)$ for all $x \in D^+$.

For $x \in D^-$, from $h(x) \leqslant \varphi(z)$ one has $\min\{h(x), g(x)\} \leqslant h(x) \leqslant \varphi(z)$ and hence $\varphi(z) \geqslant \varphi(x)$ for all $x \in D^-$.

We are now in a position to establish the main result of this section.

THEOREM 2. *Let $z \in D$ and assume there exist a $k \in I(z)$ and a $v \in \mathbb{R}^n$ s.t. $f_k(v) < f_k(z)$. Then a sufficient condition for z to be a global maximum for (PCMP) is:*

$$\partial f_k(y) \bigcap N(clco(D_k(z)), y) \neq \emptyset \text{ for all } y \text{ s.t. } f_k(y) = F(z) \qquad \text{(gS)}$$

Proof. By the sufficient part of Theorem 1 applied to v of (gNS) condition above, we have

$$f_k(z) \geqslant f_k(x) \text{ for all } x \in clco(D_k(z))$$

and hence

$$f_k(z) \geqslant f_k(x) \text{ for all } x \in D_k(z). \qquad (2)$$

Denoting $\psi_k(x) = min\{f_j(x) \mid j \in M \setminus \{k\}\}$ then $x \in D_k(z)$ implies

$$\psi_k(x) > F(z) \text{ for all } x \in D.$$

On the other hand, $F(z) = f_k(z)$ holds, since $k \in I(z)$. Therefore, (2) is equivalent to $F(z) \geqslant f_k(x)$ for all $x \in D$ such that $\psi_k(x) > F(z)$.

Finally, using Lemma 3 we get z a global maximum since

$$F(z) \geqslant F(x) = \min\{f_k(x), \psi_k(x)\} \text{ for all } x \in D.$$

REMARK 4. The assumption that there are $k \in I(z)$ and $v \in \mathbb{R}^n$ such that $f_k(v) < f_k(z)$ means that z is not a local minimum of $F(\cdot)$ in $\mathbb{R}^n$. If this assumption is violated, in other words if for all $k \in I(z)$ one has $z = \arg\min\{f_k(x) \mid x \in \mathbb{R}^n\}$, then $F(z) = f_k(z) \leqslant f_k(x) \quad (k \in I(z))$ and $F(z) < f_j(z)(j \in M \setminus I(z))$. Hence there exists some neighborhood say a ball around z of radius $\varepsilon > 0$ such that for all $x \in B(z, \varepsilon) \cap D$ we have $F(z) \leqslant F(x)$.

On the other hand, for all $k \in I(z)$ $0 \in \partial f_k(z) \cap N(clco(D_k(z), z)$. In this case, a local search can improve $F(z)$ since any feasible direction gives a better point with respect to (PCMP).

REMARK 5. The sufficient global optimality condition (gS) could be written as follows

$$\begin{cases} \text{there exist a } k \in I(z) \text{ and a } v \in \mathbb{R}^n \text{ s.t. } f_k(v) < f_k(z) \\ \text{and there also exists a } y_k^* \in \partial f_k(y) \text{ s.t. } \langle y_k^*, x - y\rangle \leqslant 0, \\ \text{for all } x \in clco(D_k(z)) \text{ and } y \text{ s.t. } f_k(y) = F(z). \end{cases}$$

REMARK 6. Let (gS) be violated at z, in other words for all $k \in I(z)$ there are y^k, u^k fullfilling respectively $f_k(y^k) = F(z)$ and $u^k \in clco(D_k(z))$ and such that for all $y_k^* \in \partial f_k(y^k)$ the inequality $0 < \langle y_k^*, u^k - y^k\rangle$ holds.

Then by Lemma 2 there exists a $w^k \in D_k(z)$ such that $\langle y_k^*, u^k\rangle \leqslant \langle y_k^*, w^k\rangle$. So, due to the convexity of all functions $f_k(\cdot)$ we have $0 < \langle y_k^*, u^k - y^k\rangle \leqslant \langle y_k^*, w^k - y^k\rangle \leqslant f_k(w^k) - f_k(y^k)$, that implies $F(z) < f_k(w^k)$. On the other hand, by definition of $D_k(z)$, $w^k \in D_k(z)$ implies $w^k \in D$ and $F(z) < f_j(w^k)$ for all $j \in M \setminus \{k\}$. As a result, we have a better point $w^k \in D$.

REMARK 7. By Proposition 1 and Lemma 2 it is easy to see that (gS) is not only a sufficient, but a necessary and sufficient condition for a global maximum of (PCMP).

EXAMPLE 2. Consider the problem in $\mathbb{R}^2$ to maximize the piecewise convex function:

$$F(x) = min\{f_j(x) \mid j = 1, 2, 3, 4, 5\},$$

where

$$f_1(x) = x_1^2 + (x_2 + 4)^2 - 36,$$
$$f_2(x) = (x_1 + 8)^2 + (x_2 - 3)^2 - 36,$$
$$f_3(x) = x_1^2 + (x_2 - 8)^2 - 16,$$
$$f_4(x) = (x_1 - 8)^2 + (x_2 - 3)^2 - 53,$$
$$f_5(x) = (x_1 - 10)^2 + (x_2 + 10)^2 - 4$$

subject to

$$D = \{x \in \mathbb{R}^2 \mid -4 \leqslant x_1 \leqslant 10, \quad -6 \leqslant x_2 \leqslant 8, \quad x_1 - x_2 \leqslant 10\}.$$

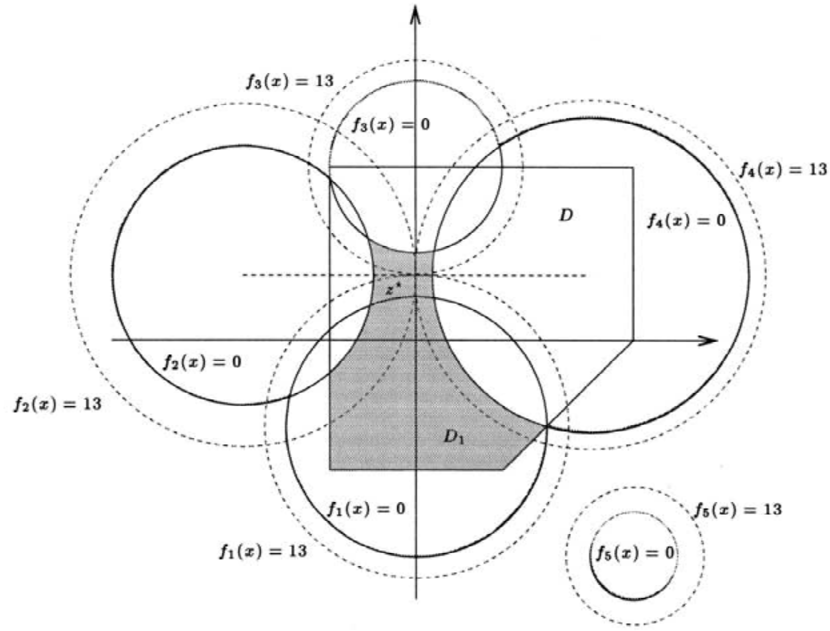

Figure 2. A simple example 2

– The point $z = (6, -4)^\top$ is a local maximum with $F(z) = 0$. One wonders if it is a global maximum.
The Lebesque set $\mathcal{L}_F(F(z))$ (where $\mathcal{L}_\phi(\alpha) = \{x \in \mathbb{R}^n \mid \phi(x) \leqslant \alpha\}$) contains two nonconnected sets and one of them is a nonconvex set (see Figure 2.).
Functions $f_1(\cdot)$ and $f_4(\cdot)$ are active at z.
Let us consider the function $f_1(\cdot)$. According to our notation

$$D_1(z) = D \cap \{x \mid f_2(x) > 0, \quad f_3(x) > 0, \quad f_4(x) > 0\}$$

since $f_5(x) > 0$ for all $x \in D$ and $F(z) = 0$.
In order to use the necessary global optimality condition (gN), one has to check the inclusion $D_1(z) \subset \mathcal{L}_{f_1}(F(z))$. It is easy to see that $D_1(z)$ is not included in $\mathcal{L}_{f_1}(F(z))$ since at $y = (0, 2)^\top$: $f_1(y) = F(z)$ and $u = (0, 3)^\top \in D_1(z)$ the inequality $\langle \nabla f_1(y), u - y\rangle > 0$ holds. Therefore z is not a global maximum.
– Now, we consider another point $z^* = (-1.08333, 2.83333)^\top$ which is also a local maximum. At the point, functions $f_2(\cdot)$, $f_3(\cdot)$ are active. And one can see that $D_2(z^*)$ is included in $\mathcal{L}_{f_2}(F(z^*))$, $(F(z^*) = 11.8681)$. That is enough, according to the sufficient global optimality conditions (gS), to say that z^* is a global maximum.

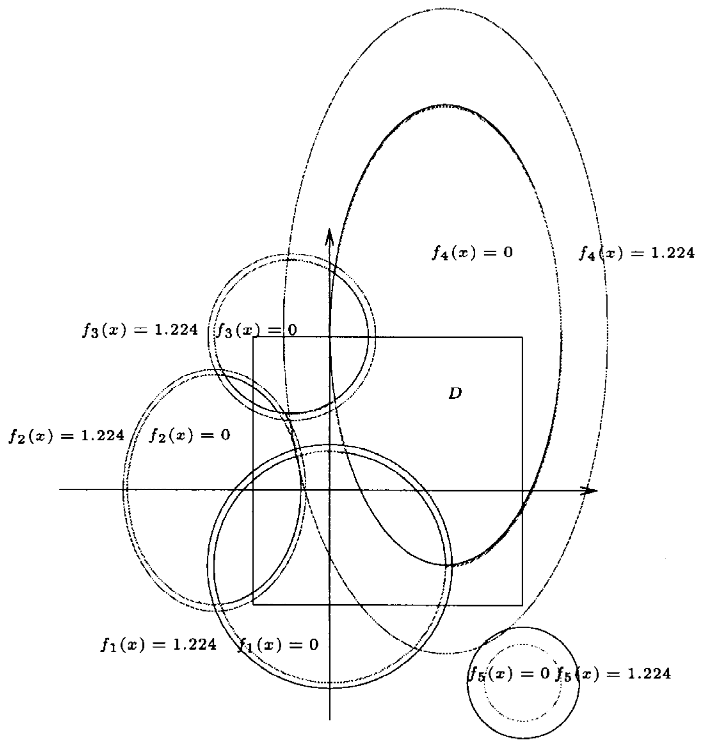

Figure 3. A non trivial example 3

EXAMPLE 3. Here we consider a piecewise convex maximization problem in $\mathbb{R}^2$ with functions (see Figure 3.)

$$
\begin{aligned}
&f_1(x) = x_1^2 + (x_2 + 2)^2 - 9,\\
&f_2(x) = 9(x_1 + 3)^2 + 4x_2^2 - 36,\\
&f_3(x) = (x_1 + 1)^2 + (x_2 - 4)^2 - 4,\\
&f_4(x) = \frac{1}{9}(x_1 - 3)^2 + \frac{1}{36}(x_2 - 4)^2 - 1,\\
&f_5(x) = (x_1 - 5)^2 + (x_2 + 5)^2 - 1
\end{aligned}
$$

subject to the box constraint:

$$D = \{x \in \mathbb{R}^2 \mid -2 \leqslant x_1 \leqslant 5, \quad -3 \leqslant x_2 \leqslant 4\}.$$

We consider it just to show some difficulty of solving (PCMP). It seems to us that even this two dimensional problem is not trivial to solve.

It is easy to see that there are a number of local maxima on the vertices, edges and interior of the box. In other words, the global maximum could be anywhere in the box.

As in Example (2) using (gN) we can escape from local maxima which are on the box vertices and edges, and look for it inside the box. There is no point in the box where all functions $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$, $f_4(\cdot)$ are active.

Here the point $(-1.3286, 1.7381)^\top$ is a global maximum with a value of $f(z) = 1.2240$.

4. A Preliminary Algorithm

The problem in proving Lemma 2 is that Caratheodory's theorem only provides the *existence* of a point in $D_k(z)$. This section is devoted to turning (gN) and (gS) into an algorithm under further weak assumptions.

Let us assume that in (1) all functions $f_j(\cdot)$ are quadratic and strongly convex. Let, furthermore, the domain set of (PCMP) be a full dimensional polytope; in other words

$$f_j(x) = \frac{1}{2}\langle Q^j x, x\rangle - \langle c^j, x\rangle, \quad j \in M,$$

$$D = \{x \in \mathbb{R}^n \mid Ax \leqslant b\},$$

where $Q^j = [Q^j]^\top$ and $Q^j > 0$ (positive definite) and $dim(D) = n$.

Let $x \in D$.

1. Let z be a local maximum of (PCMP) with the starting point x.
2. Construct $I(z)$ and choose $s \in I(z)$.
3. Approximate $D_s(z)$ by polytope $\Phi = \{x \in \mathbb{R}^n \mid Px \leqslant p\}$, where $D_s(z) \subset \Phi \subset D$.
4. **for** l=1 **to** maxiteration **do**

    $y =$ random point on level set $f_s(y) = F(z)$ not in tabu list;
    $u = \arg\max\{\langle \nabla f_s(y), x\rangle / x \in \Phi\}$; /* linearized problem */
    if $\langle \nabla f_s(y), u - y\rangle > 0$ and $u \in D_s(z)$
    then $x := u$; goto 1; /* better point */
    else
        if $u \notin D_s(z)$
        then $\Phi := \Phi \bigcap \{x \mid \langle d, x\rangle \leqslant n, \}$ /* add cutting plane */
        if $\langle \nabla f_s(y), u - y\rangle \leqslant 0$
        then add y on tabu list /* tabu point on level set */

   endfor

REMARK 8. Cutting plane
Let us compute $r = arg\min_j\{f_j(u) \mid j \in M\}$ and the index set of active constraints at u

$$J(u) = \{l \mid [Pu]_l = p_l\},$$

where P (resp. $P(u)$) denotes the matrix of constraints (resp. active constraints at u) of polytope Φ; under the full dimensionality assumption, $[P(u)]^{-1}$ definitely exists. Let V be the set (columnwise) of points on the level set $f_r(\cdot)$ intersected by the active cone, namely

$$V = u \otimes e^{\top} - [P(u)]^{-1}\alpha^r,$$

where $\otimes$ denotes Kronecker's product and ($\alpha^r \in \mathbb{R}^n_+$) solves the quadratic equations for every column vector v^i of V

$$f_r(v^i(\alpha)) = F(z^k).$$

Then vector d, found as a solution of the linear system $Vd = n$ yields a new cut for polytope Φ, as is well known in global optimization field; notice that right-hand-side introduces a normalizing factor to avoid tailing off effects since it is usual to observe such effects in similar algorithms [4, 11].

REMARK 9. Under full dimensionality of (PCMP), each level set could be partitioned through the linearized problem at y (each class having the solution u of linearized problem as a representative). This suggests introducing some tabu list throughout the random generation. Let y be a random point on level set $f_s(y) = F(z)$, then y is tabu if

$$y = [Q^s]^{-1}[c + P(u)\beta],$$

for some $\beta \in \mathbb{R}^n_+$ such that $\langle e, \beta\rangle = 1$. A non full dimensional case might require deeper insight into the projection and lifting processes to reach at the same partitioning concept!

REMARK 10. In practice we can use $D = \{x \mid Ax \leqslant b\}$ as an initial Φ.

REMARK 11. Whenever $\langle \nabla f_s(y), u - y\rangle > 0$ and $u \in D_s(z)$, we have found a better point to restart a local search, since

$$0 < \langle \nabla f_s(y), x - y\rangle \leqslant f_s(x) - f_s(y) = f_s(x) - F(z),$$

according to the convexity of $f_s(\cdot)$. Moreover, by definition of $D_s(z)$, we have $f_i(x) > F(z)$ for all $i \in M \setminus \{s\}$, which achieves $F(x) \geqslant F(z)$.

5. Concluding Remarks

In this paper we have shown how global optimality conditions for (CMP) carries over to (PCMP) and we have given a preliminary algorithm for (PCMP).

Considering the well known result [10], [14] from convex analysis:
given two convex sets $A, B \in \mathbb{R}^n (B = cl(B))$. Inclusion $A \subset B$ is true if one of the following equivalent conditions is satisfied

1. $(B - y)^\circ \in (A - y)^\circ$ for $y \in (A \cap B)$;
2. $\sigma(y \mid A) \leqslant \sigma(y \mid B)$ for all $y \in \mathbb{R}^n$;
3. $N(y \mid B) \subset N(y \mid A)$ for all $y \in bd(B)$;

our main Theorem 1, adds the further refinement to the last item

- $N(y \mid B) \bigcap N(y \mid A) \neq \emptyset$ for all $y \in bd(B)$

where $cl(D)$, D°, $\sigma(\cdot \mid D)$, $N(\cdot \mid D)$, $bd(D)$ are used for the closure, the polar, the support function, the normal cone and the boundary of D respectively.

To show the close connection with this refinement, we used in Section 2, for (CMP), $A = D$, $B = \mathcal{L}_f(f(z))$ and in Section 3, for (PCMP), $A = clco(D_k(z))$, $B = \mathcal{L}_{f_k}(F(z))$.

Acknowledgements

I am grateful to Dominique Fortin for his many helpful comments on earlier drafts of this paper. I would like to thank two anonymous referees for their valuable comments and suggestions.

References

1. Benson, H.P. (1995), *Concave Minimization: Theory, applications and algorithms*, Kluwer Academic, Dordrecht/Boston/London.
2. Dur, M., Horst, R. and Locatelli, M. (1998), Necessary and sufficient global optimality conditions for convex maximization revisited, *Journal of Mathematical Analysis and Applications* 217: 637–649.
3. Flores-Bazan, F. (1997), On minima of the difference of functions. *J. Optim. Theory Appl.* 93(3): 525–531.
4. Fortin, Dominique and Tsevendorj, Ider (2000), Global optimization and multiknapsack: a percolation algorithm. *Le rapport de recherche de l'INRIA, France* (3912): 19, Avril 2000.
5. Horst, R., Pardalos, P.M. and Van Thoai, N. (1995), *Introduction to Global Optimization*, Kluwer Academic.
6. Horst, R. and Tuy, H. (1990), *Global Optimization*, Springer-Verlag.
7. Hiriart-Urruty, J.B. (1989), *From Convex Optimization to Nonconvex Optimization, Part I. Necessary and Sufficient Conditions for Global Optimality*, Plenum, New York.
8. Hiriart-Urruty, J.B. (1995), *Conditions for Global Optimality*. Kluwer Academic, Dordrecht/Boston/London.
9. Kim, D. and Pardalos, P.M. (2000), A dynamic domain contraction algorithm for nonconvex piece-wise linear network flow problems. *J. Global Optim.* 17: 225–234.
10. Rockafellar, R.T. (1970), *Convex Analysis*. Princeton University Press.
11. Strekalovsky A.S. and Tsevendorj, I. (1998), Testing the r-strategy for a reverse convex problem. *J. Global Optim.* 13(1): 61–74.
12. Strekalovskii, A.S. (1987), On the global extremum problem. *Soviet Dokl.* 292: 1062–1066.
13. Strekalovsky, A.S. (1993), The search for a global maximum of a convex functional on an admissible set. *Computational Mathematics and Mathematical Physics* 33(3): 315–328.
14. Strekalovsky, A.S. (1998), Global optimality conditions for nonconvex optimization. *J. Global Optim.* 12(4): 415–434.

15. Tsevendorj, I. (1998), To a global solution characterization of nonconvex problems. In *11-th Baikal International School-Seminar on Optimization Methods and its applications* pp. 212–215, Irkutsk, Russia.
16. Tsevendorj, I. (1998), On the conditions for global optimality, *Journal of the Mongolian Mathematical Society* 1(2): 58–61.